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# Operator relations for SU(3) breaking contributions to K and K$^*$ distribution amplitudes

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Abstract: We derive constraints on the asymmetry $a_1$ of the momentum fractions carried by quark and antiquark in $K$ and $K^*$ mesons in leading twist. These constraints follow from exact operator identities and relate $a_1$ to SU(3) breaking quark-antiquark-gluon matrix elements which we determine from QCD sum rules. Comparing our results to determinations of $a_1$ from QCD sum rules based on correlation functions of quark currents, we find that, for $a_1^\parallel(K^*)$ the central values agree well and come with moderate errors, whereas for $a_1(K)$ and $a_1^\perp(K^*)$ the results from operator relations are consistent with those from quark current sum rules, but come with larger uncertainties. The consistency of results confirms that the QCD sum rule method is indeed suitable for the calculation of $a_1$. We conclude that the presently most accurate predictions for $a_1$ come from the direct determination from QCD sum rules based on correlation functions of quark currents and are given by:

$$a_1(K) = 0.06 \pm 0.03, \quad a_1^\parallel(K^*) = 0.03 \pm 0.02, \quad a_1^\perp(K^*) = 0.04 \pm 0.03.$$

Keywords: Phenomenological Models, Kaon Physics, Sum Rules.



 

## Contents



## 1. Introduction

Hadronic light-cone distribution amplitudes (DAs) of leading twist have been attracting considerable interest in the context of B physics. They enter the amplitudes of QCD processes that can be described in collinear factorisation, which include, to leading order in an expansion in $1/m_b$, a large class of nonleptonc B decays [1]. DAs are also an essential ingredient in the calculation of weak decay form factors such as $B \to \pi, \rho, K, K^*$ from QCD sum rules on the light-cone and the description of factorisable contributions to these form factors in SCET, see [2, 3] for recent papers. These decays, and their CP asymmetries, are currently being studied at the B factories BaBar and Belle and are expected to yield essential information about the pattern of CP violation and potential sources of flavour violation beyond the SM.

One particular problem in this context is the size of SU(3) breaking corrections to $K$ and $K^*$ DAs, which has been studied in a number of recent papers [4–7]. The DAs themselves are defined as matrix elements of quark-antiquark gauge-invariant nonlocal operators on the light-cone. To leading-twist accuracy, there are three such DAs for $K$ and $K^*$ ($z^2 = 0$):

$$\langle 0|\bar{q}(z)\not{z}\gamma_5[z,-z]s(-z)|K(q)\rangle = if_K(qz)\int_0^1 du\, e^{i\xi(qz)}\phi_K(u)\,,$$

$$\langle 0|\bar{q}(z)\not{z}[z,-z]s(-z)|K^*(q,\lambda)\rangle = (e^{(\lambda)}z)f_K^{\parallel}m_{K^*}\int_0^1 du\, e^{i\xi(qz)}\phi_K^{\parallel}(u),$$

$$\langle 0|\bar{q}(z)\sigma_{\mu\nu}[z,-z]s(-z)|K^*(q,\lambda)\rangle = i(e_\mu^{(\lambda)}q_\nu - e_\nu^{(\lambda)}q_\mu)f_K^{\perp}(\mu)\int_0^1 du\, e^{i\xi(qz)}\phi_K^{\perp}(u), \quad (1.1)$$

with the Wilson-line

$$[z,-z] = \text{Pexp}\left[ig\int_{-1}^{1} d\alpha\, z^\mu A_\mu(\alpha z)\right]$$

inserted between quark fields to render the matrix elements gauge-invariant. In the above definitions, $e_\nu^{(\lambda)}$ is the polarisation vector of a vector meson with polarisation $\lambda$; there are two leading-twist DAs for vector mesons, $\phi_K^{\parallel}$ and $\phi_K^{\perp}$, corresponding to longitudinal and transverse polarisation, respectively. The integration variable $u$ is the (longitudinal) meson momentum fraction carried by the quark, $\bar{u}\equiv 1-u$ the momentum fraction carried by the antiquark and $\xi = u-\bar{u}$. The decay constants $f_K^{(\parallel,\perp)}$ are defined in the usual way by the local limit of eqs. (1.1) and chosen in such a way that

$$\int_0^1 du\, \phi(u) = 1. \quad (1.2)$$

All three distributions $\phi_K, \phi_K^{\parallel}, \phi_K^{\perp}$ can be expanded in Gegenbauer polynomials $C_n^{3/2}$,

$$\phi(u) = 6u\bar{u}\left(1+\sum_{n\geq 1} a_n C_n^{3/2}(2u-1)\right), \quad (1.3)$$

where the $a_n$ are hadronic parameters, the so-called Gegenbauer moments.

The most relevant quantities characterising SU(3) breaking of these DAs are the decay constants $f_K$ and $f_K^{\perp,\parallel}$, and $a_1(K)$ and $a_1^{\perp,\parallel}(K^*)$, which are related to the first moment of the corresponding leading-twist DA. $a_1$ describes the difference of the average longitudinal momenta of the quark and antiquark in the two-particle Fock-state component of the meson, a quantity that vanishes for particles with equal-mass quarks (particles with definite G-parity). The decay constants $f_K$ and $f_K^{\parallel}$ can be extracted from experiment; $f_K^{\perp}$ has been calculated from both lattice [8] and QCD sum rules, e.g. ref. [7]. In this paper we focus on the determination of $a_1$: no lattice calculation of this quantity has been attempted yet, so essentially all available information on $a_1$ comes from QCD sum rule calculations. $a_1$ can be calculated either directly from the correlation function of two quark currents [4, 5, 7, 9, 10] or from operator identities relating it to certain quark-quark-gluon matrix elements, denoted $\kappa_4$, which are calculated from QCD sum rules themselves [6]. In a previous paper, ref. [7], we have obtained the following results from the first method, at the scale of 1 GeV:

$$a_1(K)^{\rm BZ} = 0.050\pm 0.025, \quad a_1^{\parallel}(K^*)^{\rm BZ} = 0.025\pm 0.015, \quad a_1^{\perp}(K^*)^{\rm BZ} = 0.04\pm 0.03, \quad (1.4)$$

whereas Braun and Lenz found the following results from operator identities [6]:

$$a_1(K)^{\rm BL} = 0.10\pm 0.12, \qquad a_1^{\parallel}(K^*)^{\rm BL} = 0.10\pm 0.07. \quad (1.5)$$

These results were obtained to first order in $m_s$ and neglecting explicit terms in $m_s^2$ and $m_q$ in the operator identities. Numerically, however, these terms are not negligible: the $O(m_s^2)$ correction shifts $a_1(K)$ by $+0.17$ and $a_1^{\parallel}(K^*)$ by $+0.08$ for our central value of $m_s$. Corrections in $m_q$ are small for $a_1^{\parallel}(K^*)$, but chirally enhanced for $a_1(K)$ and shift $a_1(K)$ by $+0.04$ for our central value of $m_q$. A consistent inclusion of $O(m_{q,s})$ effects requires the calculation of these terms also for $\kappa_4$. In the present paper, we present such a calculation and improve the sum rules for $\kappa_4$ derived in ref. [6] by the inclusion of all dominant terms to $O(m_q^2)$ and $O(m_s^2)$, which include in particular two-loop perturbative and gluon-condensate contributions. The perturbative contributions come with large coefficients and prove to be very relevant numerically. We then construct several sum rules for $\kappa_4$ which differ by the chirality structure of the involved currents and the spin-parity assignment of the hadronic states coupling to them. We provide criteria that allow one to identify the sum rules most suitable for the calculation of $\kappa_4$ and obtain the corresponding numerical results, including a careful analysis of the theoretical uncertainty of $\kappa_4$ and the corresponding values of $a_1$. One important finding of our paper is that the results of these calculations agree, within errors, with those from the quark current sum rules, which shows that the application of the QCD sum rule method to the calcualation of $a_1$ yields mutual consistent results. It is this consistency that strengthens our confidence in the validity of the results for $a_1$.

Our paper is organised as follows: in section 2 we derive the operator relations for $a_1$, in section 3 we obtain numerical results for the corresponding matrix elements and compare with the results of ref. [7]. In section 4 we summarise and conclude. The paper also contains two appendices giving explicit expressions for all relevant correlation functions and Borel transforms.

## 2. Exact identities for $a_1$

In ref. [6], the following relations were obtained for $a_1(K)$ and $a_1^{\parallel}(K^*)$:

$$\frac{9}{5}\,a_1(K) = -\frac{m_s-m_q}{m_s+m_q} + 4\,\frac{m_s^2-m_q^2}{m_K^2} - 8\kappa_4(K)\,, \tag{2.1}$$

$$\frac{3}{5}\,a_1^{\parallel}(K^*) = -\frac{f_K^{\perp}}{f_K^{\parallel}}\,\frac{m_s-m_q}{m_{K^*}} + 2\,\frac{m_s^2-m_q^2}{m_{K^*}^2} - 4\kappa_4^{\parallel}(K^*), \tag{2.2}$$

where $\kappa_4(K)$ and $\kappa_4^{\parallel}(K^*)$ are twist-4 quark-quark-gluon matrix elements defined by

$$\langle 0|\bar q(gG_{\alpha\mu})i\gamma^\mu\gamma_5 s|K(q)\rangle = iq_\alpha f_K m_K^2 \kappa_4(K), \tag{2.3}$$

$$\langle 0|\bar q(gG_{\alpha\mu})i\gamma^\mu s|K^*(q,\lambda)\rangle = e_\alpha^{(\lambda)} f_K^{\parallel} m_{K^*}^3 \kappa_4^{\parallel}(K^*). \tag{2.4}$$

$\kappa_4(K)$ and $\kappa_4^{\parallel}(K^*)$ vanish for $m_s \to m_q$ due to G-parity. The special structure of (2.1) allows one to determine the value of $\kappa_4(K)$ to leading order in $m_s$ for $m_q \to 0$ [6],

$$\kappa_4(K) = -\frac{1}{8}, \tag{2.5}$$

which is a consequence of the conservation of the axial current in the chiral limit.



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The above relations were derived from the analysis of matrix elements of the local operators ($\overleftrightarrow{D}=\overrightarrow{D}-\overleftarrow{D}$)

$$O^{(5)}_{\mu\nu} = \frac{1}{2}\,\bar{q}\gamma_\mu(\gamma_5)i\overleftrightarrow{D}_\nu\, s + \frac{1}{2}\,\bar{q}\gamma_\nu(\gamma_5)i\overleftrightarrow{D}_\mu\, s - \frac{1}{4}\,g_{\mu\nu}\bar{q}i(\gamma_5)\overleftrightarrow{\not{D}}\, s, \tag{2.6}$$

whose divergence can be expressed in terms of bilinear quark operators. In this section, we rederive these relations in a different way and obtain a new relation for $a_1^\perp(K^*)$.

The starting point for our analysis are the exact nonlocal operator relations [11, 12]

$$\begin{aligned}\frac{\partial}{\partial x_\mu}\,\bar{q}(x)\gamma_\mu(\gamma_5)s(-x) &= -i\int_{-1}^{1} dv\, v\bar{q}(x)x_\alpha gG^{\alpha\mu}(vx)\gamma_\mu(\gamma_5)s(-x)\\ &\quad -(m_s \pm m_q)\bar{q}(x)i(\gamma_5)s(-x), \qquad (2.7)\\ \partial^\mu\{\bar{q}(x)\gamma_\mu(\gamma_5)s(-x)\} &= -i\int_{-1}^{1} dv\, \bar{q}(x)x_\alpha G^{\alpha\mu}(vx)\gamma_\mu(\gamma_5)s(-x)\\ &\quad -(m_q \mp m_s)\bar{q}(x)i(\gamma_5)s(-x), \qquad (2.8)\end{aligned}$$

where the total translation $\partial_\mu$ is defined as

$$\partial_\mu\,\{\bar{q}(x)\Gamma s(-x)\} \equiv \frac{\partial}{\partial y_\mu}\,\{\bar{q}(x+y)[x+y,-x+y]\Gamma s(-x+y)\}\Big|_{y\to 0}. \tag{2.9}$$

The corresponding nonlocal matrix elements are, for $K$ and $K^*_\parallel$ ($x^2 \neq 0$):

$$\begin{aligned}\langle 0|\bar{q}(x)\gamma_\mu\gamma_5 s(-x)|K(q)\rangle &= if_K q_\mu \int_0^1 du\, e^{i\xi qx}\left[\phi_K(u) + O(x^2)\right]\\ &\quad + \frac{i}{2}f_K m_K^2 \frac{1}{qx}x_\mu \int_0^1 du\, e^{i\xi qx}[g_K(u) - \phi_K(u) + O(x^2)], \qquad (2.10)\\ \langle 0|\bar{q}(x)i\gamma_5 s(-x)|K(q)\rangle &= \frac{f_K m_K^2}{m_s+m_q}\int_0^1 du\, e^{i\xi qx}\left(\phi_K^p(u) + O(x^2)\right), \qquad (2.11)\\ \langle 0|\bar{q}(x)\gamma_\mu s(-x)|K^*(q,\lambda)\rangle &= f_K^\parallel m_{K^*}\left\{\frac{e^{(\lambda)}x}{qx}\,q_\mu \int_0^1 du\, e^{i\xi qx}\left[\phi_K^\parallel(u) + O(x^2)\right]\right.\\ &\quad + \left(e_\mu^{(\lambda)} - q_\mu\frac{e^{(\lambda)}x}{qx}\right)\int_0^1 du\, e^{i\xi qx}\left(g_K^v(u) + O(x^2)\right)\\ &\quad \left. - \frac{1}{2}x_\mu\frac{e^{(\lambda)}x}{(qx)^2}m_{K^*}^2\int_0^1 du\, e^{i\xi qx}\left[g_K^{(3)}(u) + \phi_K^\parallel(u) - 2g_K^v(u) + O(x^2)\right]\right\}. \qquad (2.12)\end{aligned}$$

In the above definitions, $\phi_K$ and $\phi_K^\parallel$ are the leading-twist DAs of $K$ and $K^*_\parallel$, respectively; all other functions are higher-twist DAs and have been extensively discussed in refs. [11–14].

$a_1(K)$, the quantity we are interested in, is related to the first moment of $\phi_K$:

$$a_1(K) = \frac{5}{3}M_1^{\phi_K}$$

with $M_1^f \equiv \int_0^1 du\,(u-\bar{u})f(u)$ being the first moment of the DA $f(u)$. Taking the matrix elements of (2.7) and (2.8) for $K$ and expanding to leading order in $x^2$ and next-to-leading order in $qx$, one obtains the exact relations

$$\begin{aligned} M_1^{\phi_K} - 2M_1^{g_K} &= -\frac{m_s - m_q}{m_s + m_q}, \\ \frac{1}{2}\left(M_1^{\phi_K} + M_1^{g_K}\right) &= -2\kappa_4(K) + M_1^{\phi_K^p}, \end{aligned} \tag{2.13}$$

from which one can determine $M_1^{\phi_K}$ once either $M_1^{\phi_K^p}$ or $M_1^{g_K}$ are known. $g_K$ is a twist-4 DA and $M_1^{g_K}$ contains quark-quark-gluon matrix elements itself, cf. refs. [12, 14], whereas $\phi_K^p$ is twist-3 and $M_1^{\phi_K^p}$ is completely determined in terms of the twist-2 DA $\phi_K$ and mass corrections. $M_1^{\phi_K^p}$ can be obtained from a second set of nonlocal operator relations involving tensor currents $\bar{q}(x)\sigma_{\mu\nu}\gamma_5 s(-x)$ or, equivalently, from the recursion relations for the moments of $\phi_K^p$ given in ref. [14]:

$$M_1^{\phi_K^p} = \frac{m_s^2 - m_q^2}{m_K^2}.$$

Solving (2.13) for $a_1(K)$, we then rederive eq. (2.1). Note that the first term on the right-hand side is rather sensitive to the value of $m_q$ and the second one to that of $m_s$.

For $K_\parallel^*$, the same method yields the equations

$$\begin{aligned} M_1^{\phi_K^\parallel} + M_1^{g_K^{(3)}} &= 2M_1^{g_K^v}, \\ M_1^{\phi_K^\parallel} - M_1^{g_K^{(3)}} &= -2\,\frac{f_K^\perp}{f_K^\parallel}\,\frac{m_s - m_q}{m_{K^*}} + 2\,\frac{m_s^2 - m_q^2}{m_{K^*}^2} - 4\kappa_4^\parallel(K^*). \end{aligned} \tag{2.14}$$

Again, $g_K^{(3)}$ is a twist-4 DA whose first moment is not known from any independent analysis, whereas $M_1^{g_K^v}$, the first moment of the twist-3 DA $g_K^v$, can be read off eq. (4.6) in ref. [13]:

$$2M_1^{g_K^v} = M_1^{\phi_K^\parallel} + \frac{f_K^\perp}{f_K^\parallel}\,\frac{m_s - m_q}{m_{K^*}}. \tag{2.15}$$

We can then solve (2.14) for $a_1^\parallel(K^*)$ and rederive eq. (2.2), the result obtained in ref. [6].

Let us now apply the same method to chiral-odd operators, with the aim of obtaining an analogous new expression for $a_1^\perp(K^*)$. The relevant nonlocal operator relations are

$$\begin{aligned} \frac{\partial}{\partial x_\mu}\bar{q}(x)\sigma_{\mu\nu}s(-x) = {}& -i\partial_\nu \bar{q}(x)s(-x) + (m_s - m_q)\bar{q}(x)\gamma_\nu s(-x) \\ & + \int_{-1}^{1} dv \bar{q}(x) g x_\alpha G^\alpha{}_\nu(vx)s(-x) - i\int_{-1}^{1} dv v \bar{q}(x) g x_\alpha G^{\alpha\mu}(vx)\sigma_{\mu\nu}s(-x), \end{aligned}$$

$$\begin{aligned} \partial^\mu\{\bar{q}(x)\sigma_{\mu\nu}s(-x)\} = {}& -i\frac{\partial}{\partial x_\nu}\,\bar{q}(x)s(-x) - (m_s + m_q)\bar{q}(x)\gamma_\nu s(-x) \\ & + \int_{-1}^{1} dv v \bar{q}(x) g x_\alpha G^\alpha{}_\nu(vx)s(-x) - i\int_{-1}^{1} dv \bar{q}(x) g x_\alpha G^{\alpha\mu}(vx)\sigma_{\mu\nu}s(-x). \end{aligned}$$



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(2.16)

These relations were first derived, without the terms in $m_s \pm m_q$, in ref. [11]; the terms in the quark masses are new.

The relevant $K^*$ matrix elements are given by [11]:

$$\langle 0|\bar{q}(x)\sigma_{\mu\nu}s(-x)|K^*(q,\lambda)\rangle = if_K^\perp \Bigg[(e_\mu^{(\lambda)}q_\nu - e_\nu^{(\lambda)}q_\mu)\int_0^1 du\, e^{i\xi qx}\Big[\phi_K^\perp(u) + O(x^2)\Big]$$
$$+ (q_\mu x_\nu - q_\nu x_\mu)\frac{e^{(\lambda)}x}{(qx)^2}m_{K^*}^2 \int_0^1 du\, e^{i\xi qx}\Big[h_K^t(u) - \frac{1}{2}\,\phi_K^\perp(u) - \frac{1}{2}\,h_K^{(3)}(u) + O(x^2)\Big]$$
$$+ \frac{1}{2}(e_\mu^{(\lambda)}x_\nu - e_\nu^{(\lambda)}x_\mu)\frac{m_{K^*}^2}{qx}\int_0^1 du\, e^{i\xi qx}\Big(h_K^{(3)}(u) - \phi_K^\perp(u) + O(x^2)\Big)\Bigg], \qquad (2.17)$$

$$\langle 0|\bar{q}(x)s(-x)|K^*(q,\lambda)\rangle =$$
$$= -i\left(f_K^\perp - f_K^\parallel\,\frac{m_s+m_q}{m_{K^*}}\right)\left(e^{(\lambda)}x\right)m_{K^*}^2\int_0^1 du\, e^{i\xi qx}\left(h_K^s(u) + O(x^2)\right), \qquad (2.18)$$

where, again, $\phi_K^\perp$ is the leading-twist DA of the transversely polarised $K^*$ and $h_K^{s,t}$ and $h_K^{(3)}$ are higher-twist DAs. In addition, we also need the following quark-quark-gluon matrix element:

$$\langle 0|\bar{q}(gG_\alpha{}^\mu)\sigma_{\beta\mu}s|K^*(q,\lambda)\rangle =$$
$$= f_K^\perp m_{K^*}^2\left\{\frac{1}{2}\,\kappa_3^\perp(K^*)(e_\alpha^{(\lambda)}q_\beta + e_\beta^{(\lambda)}q_\alpha) + \kappa_4^\perp(K^*)(e_\alpha^{(\lambda)}q_\beta - e_\beta^{(\lambda)}q_\alpha)\right\}. \qquad (2.19)$$

Here $\kappa_3^\perp(K^*)$ is a twist-3 matrix element, $\kappa_4^\perp(K^*)$ is twist-4; both are $O(m_s - m_q)$ due to G-parity.[1] Taking matrix elements of (2.16), one obtains expressions in $q_\nu$, $e_\nu^{(\lambda)}$ and $x_\nu$. To twist-4 accuracy only the former two are relevant and yield a set of four linear equations for the four first moments of $g_K^v$, $h_K^s$, $h_K^t$ and $h_K^{(3)}$:

$$\begin{aligned}
-(\kappa_3^\perp(K^*) - 2\kappa_4^\perp(K^*)) + \delta_+ M_1^{g_K^v} + M_1^{h_K^s} &= \frac{1}{2}\,M_1^{h_K^{(3)}} + \frac{1}{2}\,M_1^{\phi_K^\perp}\,,\\
\kappa_3^\perp(K^*) + 2\kappa_4^\perp(K^*) + \delta_+ M_1^{g_K^v} - M_1^{h_K^s} - \delta_+ M_1^{\phi_K^\parallel} &= \frac{1}{2}\,M_1^{h_K^{(3)}} - M_1^{h_K^t} + \frac{1}{2}\,M_1^{\phi_K^\perp}\,,\\
3\,M_1^{h_K^{(3)}} - M_1^{\phi_K^\perp} &= 2\delta_-\,,\\
M_1^{h_K^3} - 2M_1^{h_K^t} + M_1^{\phi_K^\perp} &= 0
\end{aligned} \qquad (2.20)$$

with $\delta_\pm = \frac{f_K^\parallel}{f_K^\perp}\frac{m_s \pm m_q}{m_K^*}$. The solution of that system implies

$$\delta_+ M_1^{g_K^v} = \frac{1}{6}\,\delta_- + \frac{1}{2}\,\delta_+ M_1^{\phi_K^\parallel} + \frac{1}{3}\,M_1^{\phi_K^\perp} - 2\kappa_4^\perp(K^*),$$

[1] The normalisation of $\kappa_3^\perp(K^*)$ is chosen in such a way that $\int \mathcal{D}\underline{\alpha}\mathcal{T}(\underline{\alpha}) = \kappa_3^\perp(K^*)$ for the twist-3 DA $\mathcal{T}(\underline{\alpha})$ defined in ref. [13].

| | |
|---|---|
| $\langle \bar{q}q \rangle = (-0.24 \pm 0.01)^3\,\text{GeV}^3$ | $\langle \bar{s}s \rangle = (1-\delta_3)\,\langle \bar{q}q \rangle$ |
| $\langle \bar{q}\sigma g G q \rangle = m_0^2\,\langle \bar{q}q \rangle$ | $\langle \bar{s}\sigma g G s \rangle = (1-\delta_5)\langle \bar{q}\sigma g G q \rangle$ |
| $\left\langle \frac{\alpha_s}{\pi}\,G^2 \right\rangle = (0.012 \pm 0.003)\,\text{GeV}^4$ | |
| $m_0^2 = (0.8 \pm 0.1)\,\text{GeV}^2, \quad \delta_3 = 0.2 \pm 0.2, \quad \delta_5 = 0.2 \pm 0.2$ | |
| $\overline{m}_s(2\,\text{GeV}) = (100 \pm 20)\,\text{MeV} \quad \longleftrightarrow \quad \overline{m}_s(1\,\text{GeV}) = (137 \pm 27)\,\text{MeV}$ | |
| $\overline{m}_q(\mu) = \overline{m}_s(\mu)/R, \quad R = 24.4 \pm 1.5$ | |
| $\alpha_s(m_Z) = 0.1187 \pm 0.002 \;\longleftrightarrow\; \alpha_s(1\,\text{GeV}) = 0.534^{+0.064}_{-0.052}$ | |
| $f_K = (0.160 \pm 0.002)\,\text{GeV}, \qquad f_K^{\parallel} = (0.217 \pm 0.005)\,\text{GeV}$ | |
| $f_K^{\perp} = (0.185 \pm 0.010)\,\text{GeV}$ | |

**Table 1:** Input parameters for sum rules at the renormalisation scale $\mu = 1\,\text{GeV}$. The value of $m_s$ is obtained from unquenched lattice calculations with $n_f = 2$ flavours as summarised in [15], which agrees with the results from QCD sum rule calculations [16]. $\overline{m}_q$ is taken from chiral perturbation theory [17].[2] $\alpha_s(m_Z)$ is the PDG average [19], $f_K$ and $f_K^{\parallel}$ are known from experiment and $f_K^{\perp}$ has been determined in refs. [7, 8]. The errors of quark masses and condensates are treated as correlated, see text.

which must agree with $M_1^{g_K^v}$ as given in eq. (2.15). Solving for $a_1^{\perp}(K^*)$, one finds

$$\frac{3}{5}\,a_1^{\perp}(K^*) = -\frac{f_K^{\parallel}}{f_K^{\perp}}\,\frac{m_s - m_q}{2m_{K^*}} + \frac{3}{2}\,\frac{m_s^2 - m_q^2}{m_{K^*}^2} + 6\kappa_4^{\perp}(K^*), \tag{2.21}$$

which is the wanted new relation for $a_1^{\perp}(K^*)$. Note that in all three relations (2.1), (2.2) and (2.21) $\kappa_4$ enters multiplied with a large numerical factor which implies that the theoretical uncertainty of the resulting values of $a_1$ will be much larger than that of $\kappa_4$ itself.

## 3. QCD sum rules for $\kappa_4$, $\kappa_4^{\parallel}$ and $\kappa_4^{\perp}$

In order to obtain numerical predictions for $a_1$ from the relations derived in the last section, one needs to know the values of the $\kappa_4$ matrix elements. $\kappa_4(K)$ and $\kappa_4^{\parallel}(K^*)$ have been calculated in ref. [6] from QCD sum rules to leading order in SU(3) breaking parameters with the following results:

$$\kappa_4(K)^{\text{BL}} = -0.11 \pm 0.03, \qquad \kappa_4^{\parallel}(K^*)^{\text{BL}} = -0.050 \pm 0.010\,. \tag{3.1}$$

These results refer to a renormalisation scale of 1 GeV.

[2] $m_q$ has also been determined from lattice calculations. The most recent papers on this topic are refs. [18]. The central value of $m_s/m_q$ determined in the first of these papers with $n_f = 2$ running flavours and nonperturbative renormalisation agrees with the result from chiral perturbation theory, whereas the result of the second, obtained with $n_f = 3$ and perturbative (two-loop) renormalisation, is a bit lower. As the field appears to develop rapidly, we refrain from taking either side and stay with the result from chiral perturbation theory.

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In this section we present QCD sum rules for $\kappa_4(K)$ and $\kappa_4^{\parallel}(K^*)$ which are accurate to NLO in SU(3) breaking and also a new sum rule for $\kappa_4^{\perp}(K^*)$ to the same accuracy. For all sum rules we include $O(m_q)$ effects. The sum rules are of the generic form

$$\kappa_4(K) f_K^2 m_K^n e^{-m_K^2/M^2} + \text{contribution from higher mass states} = \mathcal{B}_{M^2}\Pi_G, \tag{3.2}$$

and correspondingly for $K^*$. $\Pi_G$ are correlation functions of type

$$\Pi_G(q) = i\int d^4y e^{iqy}\langle 0|T[\bar{q}(gG_{\alpha\mu})\Gamma_1^{\mu}s](y)[\bar{s}\Gamma_2 q](0)|0\rangle$$

with suitably chosen Dirac structures $\Gamma_1^{\mu}$ and $\Gamma_2$; explicit expressions for all relevant $\Pi_G$ are given in appendix A. $\mathcal{B}_{M^2}\Pi_G$ is the Borel transform of $\Pi_G$, $M^2$ the Borel parameter and $n$ is either 2 or 4. In order to separate the ground state from higher mass contributions, one usually models the latter, using global quark hadron duality, by an integral over the perturbative spectral density:

$$\text{contribution from higher mass states} \approx \int_{s_0}^{\infty} e^{-s/M^2}\,\frac{1}{\pi}\,\text{Im}\,\Pi_G(s); \tag{3.3}$$

the parameter $s_0$ is called continuum threshold. The input parameters for the QCD sum rules are collected in table 1.

All $\kappa_4$ parameters can actually be determined from more than one sum rule derived from various $\Pi_G$ which can be characterised by the following features:

- the currents can have the same or different chirality, which results in chiral-even and chiral-odd sum rules, respectively;
- the hadronic states saturating $\Pi_G$ can have unique spin-parity or come with different parity (e.g. $0^-$ and $1^+$), which results in pure-parity and mixed-parity sum rules, respectively.

Note that all chiral-odd sum rules are also pure-parity.

In chiral-odd sum rules the quark condensates always appear in the combination $\langle\bar{q}q\rangle - \langle\bar{s}s\rangle = \delta_3\langle\bar{q}q\rangle$ and $\langle\bar{q}\sigma gGq\rangle - \langle\bar{s}\sigma gGs\rangle = \delta_5\langle\bar{q}\sigma gGq\rangle$, which induces a large dependence on the only poorly constrained parameters $\delta_{3,5}$ and also increases the impact of the gluon condensate contribution which is equally poorly known. We therefore decide to drop all chiral-odd sum rules and only use chiral-even ones.

As for mixed and pure-parity sum rules, they come with different mass dimensions: $n = 2$ in (3.2) for mixed-parity vs. $n = 4$ for pure-parity sum rules. It is an important result of this paper that the continuum contributions to the mixed-parity sum rules, for typical Borel parameters $M^2$ around $1.7\,\text{GeV}^2$, are small and below 10% for all three $\kappa_4$. Pure-parity sum rules, on the other hand, have a large continuum contribution around 30%. There are two reasons for this result: first, the additional power of $m_K^2$ in pure-parity sum rules counteracts the exponential suppression of the continuum contribution. Second, the contributions of particles with different parity have different sign: it was already found in ref. [6] that $\kappa_4(K)$ and $\kappa_4^{\parallel}(K_1)$ have opposite sign; we find that the same applies to $\kappa_4^{\parallel}(K^*)$

and the corresponding $\kappa_4(K_0^*)$ of the lowest scalar resonance, and ditto to $\kappa_4^\perp(K^*)$ and the coupling $\kappa_4^\perp(K_1)$ of the axial vector $K_1$ meson. These results suggest that the $\kappa_4$ matrix elements of opposite-parity mesons have generically different signs and tend to cancel each other in mixed-parity sum rules, which results in a small continuum contribution. From a more formal point of view it is rather obvious from the definitions eqs. (2.3), (2.4) and (2.19) that the sign of $\kappa_4$ changes under a parity transformation,[3] which is in line with our findings.

The mixed-parity sum rules for $K$ and $K^*$ do involve the three spin-parity systems $(0^-,1^+)$, $(1^-,0^+)$ and $(1^-,1^+)$. Note that for all of them the "wrong"-parity ground state (e.g. the scalar $K_0^*(1430)$) and the first orbital excitation of the "right"-parity state (e.g. the vector $K^*(1410)$) have nearly equal mass, which makes the cancellation very effective. We conclude that mixed-parity sum rules are more reliable than pure-parity ones and, as a consequence, will not consider the latter in this paper. In view of the cancellation of contributions of different sign we also decide to include explicitly only the lowest-mass ground state in the mixed-parity sum rules, which differs from the procedure adopted by the authors of ref. [6].

Let us now turn to the question how to choose the Borel parameter $M^2$ and the continuum threshold $s_0$, the internal sum rule parameters. As mentioned before, the dependence of the sum rules on $s_0$ is weak and so we simply use the same values of $s_0$ as for the quark current sum rules, i.e. $s_0(K) = (1.1 \pm 0.3)\,\mathrm{GeV}^2$, $s_0^\parallel(K^*) = (1.7 \pm 0.3)\,\mathrm{GeV}^2$ and $s_0^\perp(K^*) = (1.3 \pm 0.3)\,\mathrm{GeV}^2$ [7]. The small dependence on $s_0$ also allows one to use slightly higher values of $M^2$ than the usual 1 to $2\,\mathrm{GeV}^2$, which improves the convergence of the operator product expansion of the correlation functions and reduces the variation of the sum rule with $M^2$. We choose $M^2 = (1.6 \pm 0.4)\,\mathrm{GeV}^2$ for $K$ and $M^2 = (1.8 \pm 0.4)\,\mathrm{GeV}^2$ for $K^*$.

After this general discussion of the choice of sum rules and parameters let us now turn to the three $\kappa_4$ parameters in turn.

### 3.1 $\kappa_4(K)$

The mixed-parity sum rule for $\kappa_4(K)$ is obtained from the correlation function $\Pi_{G,2}^{(a)}$ in appendix A, eq. (A.6), and given by

$$
\begin{aligned}
f_K^2 m_K^2 \kappa_4(K)\, e^{-m_K^2/M^2} = {} & \frac{\alpha_s}{72\pi^3}\,(m_s^2 - m_q^2) \int_0^{s_0} ds\, e^{-s/M^2} \left( 10 \ln \frac{s}{\mu^2} - 25 \right) \\
& + \frac{2}{9}\,\frac{\alpha_s}{\pi}\,(m_s\langle \bar q q\rangle - m_q \langle \bar s s\rangle) \left\{ -\frac{1}{3} + \gamma_E - \ln\frac{M^2}{\mu^2} + \int_{s_0}^\infty \frac{ds}{s}\, e^{-s/M^2} \right\} \\
& + \frac{10}{9}\,\frac{\alpha_s}{\pi}\,(m_s\langle \bar s s\rangle - m_q\langle \bar q q\rangle) + \frac{1}{6M^2}\,(m_s\langle \bar s \sigma g G s\rangle - m_q \langle \bar q \sigma g G q\rangle) \\
& + \frac{m_s^2 - m_q^2}{6M^2} \left\langle \frac{\alpha_s}{\pi} G^2 \right\rangle \left\{ 1 - \frac{1}{2}\left( \ln\frac{M^2}{\mu^2} - \gamma_E + 1 \right) - M^2 \int_{s_0}^\infty \frac{ds}{2s^2}\, e^{-s/M^2} \right\} \\
& + \frac{8\pi\alpha_s}{27M^2}\,[\langle \bar q q\rangle^2 - \langle \bar s s\rangle^2]\,. \qquad (3.4)
\end{aligned}
$$

[3] In QCD parity is not a symmetry of the hadronic spectrum because the $\mathrm{U}(1)_\mathrm{A}$-symmetry is broken.

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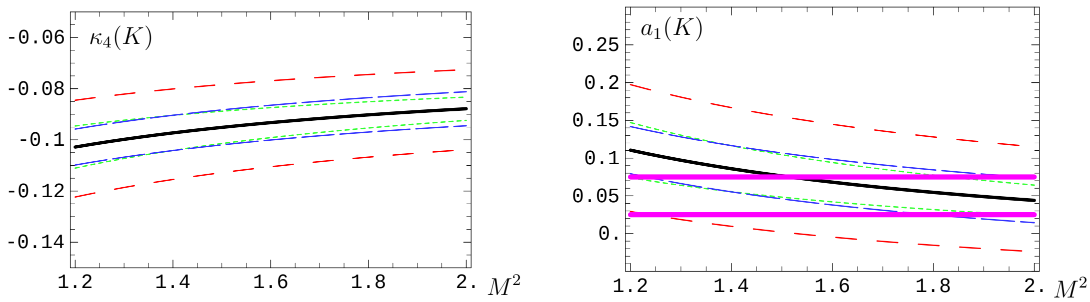


**Figure 1:** (Colour online) Left panel: $\kappa_4(K)$ from (3.4) as function of the Borel parameter $M^2$. Parameters: renormalisation scale $\mu = 1\,\text{GeV}$, $s_0 = 1.1\,\text{GeV}^2$. Solid line: central value of input parameters; dashed lines: red: $\delta_3 = 0, 0.4$, green: $\delta_5 = 0, 0.4$, blue: $\alpha_s(m_Z) = 0.1167, 0.1207$. Right panel: $a_1(K)$ as function of $M^2$ from the operator relation (2.1) (colour-coded as in the left panel) and the value of $a_1(K)$ determined in ref. [7] (purple lines).

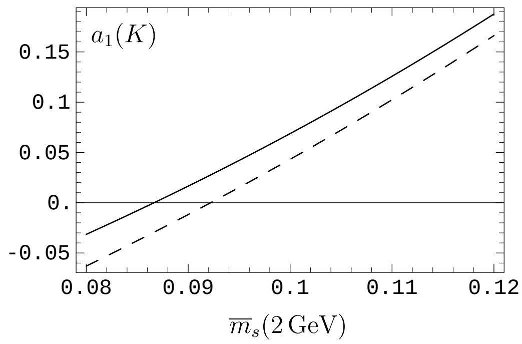


**Figure 2:** Dependence of the central value of $a_1(K)$ from (2.1) on $\overline{m}_s(2\,\text{GeV})$. Solid line: $\overline{m}_q(2\,\text{GeV}) = 4\,\text{MeV}$, dashed line: $\overline{m}_q(2\,\text{GeV}) = 0\,\text{MeV}$.

This sum rule includes all relevant contributions up to dimension six. Numerically, all dominant contributions have the same sign, with the largest one from $\langle \bar{s}s \rangle$, followed by the ones from $\langle \bar{s}\sigma gGs \rangle$ and perturbation theory which are roughly of the same size.

In figure 1 we plot the resulting values for $\kappa_4(K)$ and, via (2.1), $a_1(K)$, displaying, for illustration, explicitly the dependence on $\alpha_s$ and $\delta_{3,5}$. It is evident that the dependence of both quantities on $\delta_3$ and $\delta_5$ is nonnegligible; at the same time, the comparison with $a_1(K)$ obtained in ref. [7] from a QCD sum rule for quark currents shows that both sum rules agree within errors.[4] Note that the inclusion of the perturbative contribution is crucial: without it, we would have obtained a *negative* result for $a_1(K)$. The impact of nonzero $m_q$ is also relevant and shifts the central value of $a_1(K)$ by $+0.025$.

As for the theoretical uncertainties of $\kappa_4(K)$ and $a_1(K)$ we note that they arise first from the QCD sum rule parameters and second from the uncertainties of the hadronic

[4]The results from the quark current sum rules quoted in this paper are slightly larger than the ones given in ref. [7]. This is due to the fact that we have included infrared sensitive terms of type $m_s^2 \ln(M^2/m_s^2)$ in the contribution of the gluon condensate in the mixed quark-quark-gluon condensate rather than in the Wilson-coefficient of the gluon condensate, cf. the discussion in appendix A and ref. [20].



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parameters given in table 1. As for the former, as already stated above, we choose $M^2 = (1.6 \pm 0.4)\,\mathrm{GeV}^2$ and $s_0 = (1.1 \pm 0.3)\,\mathrm{GeV}^2$ and add the corresponding uncertainties in quadrature. As for the latter, we treat $m_q$, $m_s$, $\langle \bar{q}q \rangle$, $\langle \bar{q}\sigma g G q \rangle$, $\delta_3$ and $\delta_5$ as parameters with correlated errors. Chiral perturbation theory helps to unravel some of these correlations: for instance, one has $(m_s + m_q)/(2m_q) = m_K^2/m_\pi^2$ and $m_K^2 = -2(m_s + m_q)\langle \bar{q}q \rangle / f_\pi^2$ in LO chiral perturbation theory [17]. The dependence of $\delta_{3,5}$ on $m_s$ is unfortunately unknown (and indeed would deserve further study). In order to estimate the uncertainty of $\kappa_4(K)$ and $a_1(K)$, we hence eliminate, using the above relations, $m_q$ and $\langle \bar{q}q \rangle$ as independent parameters in favour of $m_s$, but keep $m_0^2 = \langle \bar{q}\sigma g G q \rangle / \langle \bar{q}q \rangle$ and $\delta_{3,5}$. This procedure is likely to overestimate the uncertainties induced by $\langle \bar{s}s \rangle$ and $\langle \bar{s}\sigma g G s \rangle$, but it is difficult to do better at present. Varying all remaining independent input parameters within their respective ranges given in table 1, we obtain the following results:

$$\begin{aligned} \kappa_4(K) &= -0.09 \pm 0.01 \pm 0.01 \pm 0.01 \pm 0.02 \pm 0.01 \pm 0.00 = -0.09 \pm 0.01 \pm 0.02, \\ a_1(K) &= \phantom{-}0.07 \pm 0.04 \pm 0.03 \pm 0.11 \pm 0.07 \pm 0.03 \pm 0.01 = \phantom{-}0.07 \pm 0.04 \pm 0.14, \end{aligned} \tag{3.5}$$

where the first uncertainty comes from the variation of the sum rule specific parameters $M^2$ and $s_0$, the second one from $\alpha_s$, the 3rd from $m_s$, the 4th from $\delta_3$, the 5th from $\delta_5$ and the 6th from $m_0^2 = \langle \bar{q}\sigma g G q \rangle / \langle \bar{q}q \rangle$. For the total uncertainty we give two terms: the first comes from the sum rule parameters and the second is obtained by adding all hadronic uncertainties in quadrature. As mentioned before, any uncertainty of $\kappa_4(K)$ induces a corresponding uncertainty in $a_1(K)$ that is about four times larger, except for the strange quark masses whose uncertainty also plays in the second term on the right-hand side of (2.1). The dependence of $a_1(K)$ on $m_s$ is shown in figure 2. The effect of nonzero $m_q$ in the first term on the right-hand side of (2.1) is a shift by $+0.04$, which is partially, but not completely, compensated by the $m_q$-dependent contributions to $\kappa_4(K)$. Comparing with the value of $a_1(K)$ quoted in ref. [6], eq. (1.5), we see that the central value in (3.5) is smaller and also the total uncertainty is larger. The larger error is due to the fact that we have chosen slightly larger errors for $m_s$ and also have included the uncertainty induced by $\alpha_s$.

Let us now compare the result (3.5) with the one obtained from quark current sum rules [7], with the same sequence of errors as in (3.5):

$$a_1(K)^{\rm BZ} = 0.06 \pm 0.01 \pm 0.00 \pm 0.01 \pm 0.01 \pm 0.01 \pm 0.00 = 0.06 \pm 0.01 \pm 0.02\,. \tag{3.6}$$

This number is slightly larger than the one quoted in ref. [7], cf. footnote 2. Although the central values of $a_1(K)$ agree very well and hence confirm the consistency of the sum rule results, it is obvious that the operator relation (2.1) cannot match the accuracy of the quark current sum rule and is hence not very useful for constraining $a_1(K)$.

### 3.2 $\kappa_4^\parallel(K^*)$

Let us now turn to $\kappa_4^\parallel(K^*)$. The mixed-parity sum rule is derived from the correlation function $\Pi_{G,2}^{(v)}$ in appendix A, eq. (A.11), and reads

$$\kappa_4^\parallel(K^*)(f_K^\parallel)^2 m_{K^*}^2 e^{-m_{K^*}^2/M^2} = (m_s^2 - m_q^2)\,\frac{\alpha_s}{72\pi^3}\int_0^{s_0} ds\, e^{-s/M^2}\left(10\ln\frac{s}{\mu^2} - 25\right)$$

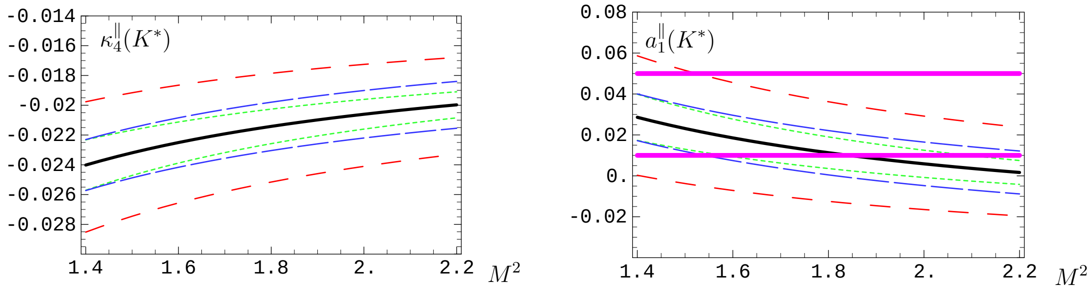


**Figure 3:** Left panel: $\kappa_4^{\parallel}(K^*)$ from (3.7) as function of the Borel parameter $M^2$. Parameters: renormalisation scale $\mu = 1\,\text{GeV}$, $s_0 = 1.7\,\text{GeV}^2$. Solid black line: central values of parameters; the coloured lines have the same meaning as in figure 1. Right panel: $a_1^{\parallel}(K^*)$ as function of $M^2$ from the operator relation (2.2) and the sum rule for $a_1^{\parallel}(K^*)$ calculated in ref. [7] (purple lines).

$$
\begin{aligned}
&- \frac{2\alpha_s}{9\pi}\left(m_s\langle\bar{q}q\rangle - m_q\langle\bar{s}s\rangle\right)\left\{-\frac{1}{3} + \gamma_E - \ln\frac{M^2}{\mu^2} + \int_{s_0}^{\infty}\frac{ds}{s}\,e^{-s/M^2}\right\}\\
&+ \frac{10\alpha_s}{9\pi}\left(m_s\langle\bar{s}s\rangle - m_q\langle\bar{q}q\rangle\right) + \frac{1}{6M^2}\left(m_s\langle\bar{s}\sigma gGs\rangle - m_q\langle\bar{q}\sigma gGq\rangle\right)\\
&+ \frac{m_s^2 - m_q^2}{6M^2}\left\langle\frac{\alpha_s}{\pi}G^2\right\rangle\left\{1 - \frac{1}{2}\left(\ln\frac{M^2}{\mu^2} - \gamma_E + 1\right) - M^2\int_{s_0}^{\infty}\frac{ds}{2s^2}\,e^{-s/M^2}\right\}\\
&+ \frac{8\pi\alpha_s}{27M^2}\left[\langle\bar{q}q\rangle^2 - \langle\bar{s}s\rangle^2\right]. \qquad (3.7)
\end{aligned}
$$

The resulting values of $\kappa_4^{\parallel}(K^*)$ and $a_1^{\parallel}(K^*)$ are shown in figure 3. Again, the contribution from perturbation theory is crucial numerically: without it, the resulting values of $a_1^{\parallel}(K^*)$ would have been negative. Our final results are:

$$
\begin{aligned}
\kappa_4^{\parallel}(K^*) &= -0.022 \pm 0.003 \pm 0.001 \pm 0.003 \pm 0.004 \pm 0.001 \pm 0.001\\
&= -0.022 \pm 0.003 \pm 0.005,\\
a_1^{\parallel}(K^*) &= \phantom{-}0.01 \pm 0.02 \pm 0.01 \pm 0.01 \pm 0.02 \pm 0.01 \pm 0.00\\
&= \phantom{-}0.01 \pm 0.02 \pm 0.03 \qquad (3.8)
\end{aligned}
$$

with the same assignment and treatment of uncertainties as in (3.5); the uncertainty coming from $f_K^{\perp}$ is included in that from $m_s$. In contrast to the pseudoscalar case, the translation of $\kappa_4^{\parallel}(K^*)$ into $a_1^{\parallel}(K^*)$ does not increase the uncertainty from $m_s$ any more than the other uncertainties, so that the total error of $a_1^{\parallel}(K^*)$ is smaller than that of $a_1(K)$. The impact of $m_q$-dependent terms in negligible. The results (3.8) differ from those of ref. [6] where the pure-parity sum rule has been used instead. The result from the quark current sum rule is

$$
a_1^{\parallel}(K^*)^{\mathrm{BZ}} = 0.03 \pm 0.02. \qquad (3.9)
$$

Again we find agreement between the results for $a_1$ from the sum rules for $\kappa_4$ and the quark current sum rules, but at the same time the uncertainty of the former is larger than that of the latter.

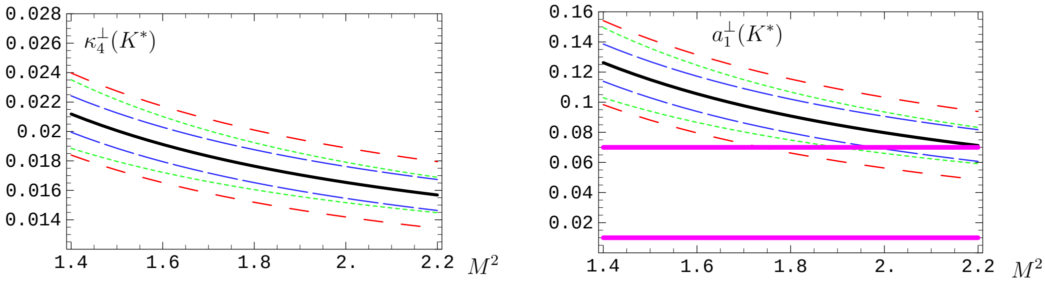


**Figure 4:** Left panel: $\kappa_4^\perp(K^*)$ from (3.10) as function of the Borel parameter $M^2$. Parameters: renormalisation scale $\mu = 1\,\text{GeV}$, $s_0 = 1.3\,\text{GeV}^2$. Solid black line: central values of parameters; the coloured lines have the same meaning as in figure 1. Right panel: $a_1^\perp(K^*)$ as function of $M^2$ from the operator relation (2.21) and the sum rule for $a_1^\perp(K^*)$ calculated in ref. [7] (purple lines).

### 3.3 $\kappa_4^\perp(K^*)$

The last parameter left to be determined is $\kappa_4^\perp(K^*)$. Its mixed-parity sum rule is derived from the correlation function $\Pi_{G,4}$, eq. (A.16), and reads

$$\begin{aligned}
\kappa_4^\perp(K^*)(f_K^\perp)^2 m_{K^*}^2 e^{-m_{K^*}^2/M^2} &= (m_s^2 - m_q^2)\,\frac{\alpha_s}{72\pi^3}\int_0^{s_0} ds\, e^{-s/M^2}\left(-6\ln\frac{s}{\mu^2} + 14\right)\\
&+ \frac{m_s\alpha_s}{3\pi}\left\{\frac{1}{3}\langle\bar q q\rangle - 2\langle\bar s s\rangle\right\} - \frac{m_q\alpha_s}{3\pi}\left\{\frac{1}{3}\langle\bar s s\rangle - 2\langle\bar q q\rangle\right\} + \frac{1}{6M^2}\left(m_q\langle\bar q\sigma gGq\rangle - m_s\langle\bar s\sigma gGs\rangle\right)\\
&+ \frac{m_s^2 - m_q^2}{12M^2}\left\langle\frac{\alpha_s}{\pi}G^2\right\rangle\left\{-2 + \left(\ln\frac{M^2}{\mu^2} - \gamma_E + 1\right) + M^2\int_{s_0}^{\infty}\frac{ds}{s^2}\,e^{-s/M^2}\right\}. \qquad (3.10)
\end{aligned}$$

The results for $\kappa_4^\perp(K^*)$ and $a_1^\perp(K^*)$ are shown in figure 4; including uncertainties, we find

$$\begin{aligned}
\kappa_4^\perp(K^*) &= 0.018 \pm 0.004 \pm 0.001 \pm 0.002 \pm 0.002 \pm 0.002 \pm 0.001\\
&= 0.018 \pm 0.004 \pm 0.004\,,\\
a_1^\perp(K^*) &= 0.09 \pm 0.04 \pm 0.01 \pm 0.01 \pm 0.02 \pm 0.02 \pm 0.01\\
&= 0.09 \pm 0.04 \pm 0.03\,. \qquad (3.11)
\end{aligned}$$

Note that the "enhancement" factor of uncertainties of $a_1^\perp(K^*)$ due to $\kappa_4^\perp(K^*)$ is 10, which is the reason for the large total uncertainty in (3.11). The impact of $m_q$-dependent terms is again negligible. The quark current sum rule yields [7]

$$a_1^\perp(K^*)^{\rm BZ} = 0.04 \pm 0.01 \pm 0.01 \pm 0.01 \pm 0.01 \pm 0.00 \pm 0.00 = 0.04 \pm 0.01 \pm 0.02. \qquad (3.12)$$

Hence, also for $a_1^\perp(K^*)$ do the results of the two approaches agree within errors, with the quark current sum rule being more accurate.

## 4. Summary and conclusions

In this paper, we have obtained the following relations for the first Gegenbauer moments of the leading-twist distribution amplitudes of $K$ and $K^*$ mesons:

$$
\begin{aligned}
\frac{9}{5}\, a_1(K) &= -\frac{m_s - m_q}{m_s + m_q} + 4\,\frac{m_s^2 - m_q^2}{m_K^2} - 8\kappa_4(K),\\
\frac{3}{5}\, a_1^{\parallel}(K^*) &= -\frac{f_K^{\perp}}{f_K^{\parallel}}\,\frac{m_s - m_q}{m_{K^*}} + 2\,\frac{m_s^2 - m_q^2}{m_{K^*}^2} - 4\kappa_4^{\parallel}(K^*),\\
\frac{3}{5}\, a_1^{\perp}(K^*) &= -\frac{f_K^{\parallel}}{f_K^{\perp}}\,\frac{m_s - m_q}{2m_{K^*}} + \frac{3}{2}\,\frac{m_s^2 - m_q^2}{m_{K^*}^2} + 6\kappa_4^{\perp}(K^*),
\end{aligned}
\tag{4.1}
$$

where the $\kappa_4$ matrix elements are defined as

$$
\begin{aligned}
\langle 0|\bar{q}(gG_{\alpha\mu})i\gamma^{\mu}\gamma_5 s|K(q)\rangle &= iq_\alpha f_K m_K^2 \kappa_4(K)\,,\\
\langle 0|\bar{q}(gG_{\alpha\mu})i\gamma^{\mu} s|K^*(q,\lambda)\rangle &= e_\alpha^{(\lambda)} f_K^{\parallel} m_{K^*}^3 \kappa_4^{\parallel}(K^*)\,,\\
\langle 0|\bar{q}(gG_{\alpha}{}^{\mu})\sigma_{\beta\mu} s|K^*(q,\lambda)\rangle &= \\
&= f_K^{\perp} m_{K^*}^2 \left\{ \frac{1}{2}\,\kappa_3^{\perp}(K^*)(e_\alpha^{(\lambda)} q_\beta + e_\beta^{(\lambda)} q_\alpha) + \kappa_4^{\perp}(K^*)(e_\alpha^{(\lambda)} q_\beta - e_\beta^{(\lambda)} q_\alpha)\right\}.
\end{aligned}
$$

The first two relations in (4.1) were already derived in ref. [6], the third is new. We have interpreted these relations as constraints on $a_1$ and calculated the three $\kappa_4$ parameters from QCD sum rules. We have improved the sum rules given in ref. [6] for $\kappa_4(K)$ and $\kappa_4^{\parallel}(K^*)$ by including two-loop perturbative contributions, the gluon condensate contribution and terms in $m_q$; the former proved to be very relevant numerically, the terms in $m_q$ are relevant for $a_1(K)$. We have also derived a new sum rule for $\kappa_4^{\perp}(K^*)$ to the same accuracy. All these sum rules exhibit only a small continuum contribution and all relevant contributions come with equal sign. The results for $a_1$ obtained from the relations (4.1) agree, within errors, with those obtained in ref. [7] from quark current sum rules which is an important confirmation of the consistency of QCD sum rule calculations of these quantities and strengthens our confidence in the results. From a phenomenological point of view, however, the operator relations (4.1) are, at least at present, less useful than the quark current sum rules for $a_1$, as the uncertainties of the $\kappa_4$ parameters are too large to allow an accurate determination of $a_1$. The uncertainties of $\kappa_4$ arise from (a) the dependence of the sum rule on the sum rule internal parameters $M^2$ and $s_0$, (b) the uncertainties of $\alpha_s$ at the hadronic scale $\sim 1\,\text{GeV}$ and (c) the uncertainties of $m_s$ and the SU(3) breaking of quark and mixed condensates parametrised by $\delta_{3,5}$. All these uncertainties enter $a_1$ multiplied by large factors 5 to 10, eqs. (4.1). In contrast, the quark current sum rules for $a_1$ studied in refs. [5, 7] are not very sensitive to these effects and come with smaller uncertainties. We hence suggest that the relations (4.1) be interpreted as constraints on $\kappa_4$ rather than $a_1$. Using the updated values of $a_1$ from quark current sum rules quoted in section 3, adding the errors linearly,

$$
a_1(K)^{\rm BZ} = 0.06 \pm 0.03, \quad a_1^{\parallel}(K^*)^{\rm BZ} = 0.03 \pm 0.02 \quad a_1^{\perp}(K^*)^{\rm BZ} = 0.04 \pm 0.03, \tag{4.2}
$$

we find by solving (4.1) for $\kappa_4$:

$$
\kappa_4(K) = -0.09 \pm 0.02,
$$

$$\kappa_4^{\parallel}(K^*) = -0.024 \pm 0.003,$$
$$\kappa_4^{\perp}(K^*) = 0.012 \pm 0.004.$$

For $\kappa_4(K)$ and $\kappa_4^{\parallel}(K^*)$ the central value agrees well with the results from the direct calculation, for $\kappa_4^{\perp}(K^*)$ there is agreement within errors. How can these results be improved? The quark current results for $a_1$ would profit from a calculation of perturbative radiative corrections $\sim m_s^2 \alpha_s$, which is technically feasible, but beyond the scope of this paper. Both $a_1$ and $\kappa_4$ would benefit from a reduction of the errors of $m_s$.

In summary, we hope that the present paper helps to settle the controversy about $a_1$ which started from the observation that the original calculation of ref. [9] suffers from a sign-mistake of the perturbative contribution, which was corrected in ref. [4]. Unfortunately, the chiral-odd sum rules used in ref. [4] come with large cancellations of the dominant contributions and are hence not very useful for precise calculations of $a_1$. In ref. [5], $a_1(K)$ was then determined from chiral-even quark current sum rules and in ref. [7] also $a_1^{(\perp,\parallel)}(K^*)$ was calculated using that method. These sum rules do not exhibit any cancellations of large contributions and are stable under the variation of all input parameters. As we have shown in this paper, these results agree with those from the operator relations (4.1) within errors, but are more accurate. We conclude that the quark current sum rule results (4.2) present the presently best determination of $a_1$. Given the phenomenological importance of $a_1$, an independent calculation on the lattice would be both timely and useful and we would like to appeal to the lattice community to take up the challenge.

## Acknowledgments

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## A. Correlation functions

In this appendix we give the relevant formulas for the correlation functions from which the QCD sum rules given in section 3 are obtained. The correlation functions are of the generic form

$$\Pi_{\alpha\ldots}(q) = i \int d^4y e^{iqy} \langle 0|T[\bar{q}(gG_{\alpha\mu})\Gamma_1^{\mu} s](y)[\bar{s}\Gamma_2 q](0)|0\rangle\,, \tag{A.1}$$

where $\Gamma_1^{\mu}$ and $\Gamma_2$ are suitably chosen Dirac structures. The dots stand for additional indices from $\Gamma_2$.

### A.1 $\kappa_4(K)$

$\kappa_4(K)$ can be extracted from either a pure-partity sum rule, to which only pseudoscalar states contribute, or a mixed-parity sum rule which also contains contributions from axialvector mesons. As for pure-parity sum rules, one possible choice of the Dirac structures is $\Gamma_1^{\mu} = i\gamma^{\mu}\gamma_5$ and $\Gamma_2 = i\gamma_5$, which results in the correlation function

$$\Pi_{\alpha}(q) = iq_{\alpha}\Pi_G^{(p)}(q^2). \tag{A.2}$$

Another choice is $\Gamma_1^\mu = i\gamma^\mu\gamma_5$ as before and $\Gamma_2 = \gamma_\beta\gamma_5$, with the correlation function

$$\Pi_{\alpha\beta}(q) = g_{\alpha\beta}\Pi^{(a)}_{G,1}(q^2) + q_\alpha q_\beta \Pi^{(a)}_{G,2}(q^2), \tag{A.3}$$

where $\Pi^{(a)}_{G,1}$ receives contributions from $1^+$ intermediate states only, whereas $\Pi^{(a)}_{G,2}$ is a mixed-parity correlation function with contributions from both $0^-$ and $1^+$ states.

These three correlation functions are not independent of each other, but related by $\partial^\beta \bar{s}\gamma_\beta\gamma_5 q = (m_s + m_q)\bar{s}i\gamma_5 q$, so that

$$\Pi^{(a)}_{G,1}(q^2) + q^2\Pi^{(a)}_{G,2}(q^2) = (m_s + m_q)\Pi^{(p)}_G + \text{ contact terms}, \tag{A.4}$$

where the contact terms are independent of $q^2$. As terms in $m_q$ are numerically relevant in the operator relation (2.1), we calculate the correlation functions to $O(m_q)$ and find

$$\begin{aligned}
\Pi^{(p)}_G(q^2) &= -(m_s - m_q)\,\frac{\alpha_s}{48\pi^3}q^2\left[\ln^2\frac{-q^2}{\mu^2} - \ln\frac{-q^2}{\mu^2}\right] - \frac{1}{4q^2}\left[\langle\bar{q}\sigma gGq\rangle - \langle\bar{s}\sigma gGs\rangle\right] \\
&\quad - \frac{\alpha_s}{3\pi}\left[\langle\bar{q}q\rangle - \langle\bar{s}s\rangle\right]\ln\frac{-q^2}{\mu^2} \\
&\quad + \frac{1}{8q^2}\left\langle\frac{\alpha_s}{\pi}G^2\right\rangle\left[m_s\left(1 - \ln\frac{-q^2}{m_s^2}\right) - m_q\left(1 - \ln\frac{-q^2}{m_q^2}\right)\right], \\
\Pi^{(a)}_{G,1}(q^2) &= (m_s^2 - m_q^2)\,\frac{\alpha_s}{144\pi^3}q^2\left[7\ln^2\frac{-q^2}{\mu^2} - 47\ln\frac{-q^2}{\mu^2}\right] \\
&\quad - \frac{m_s\alpha_s}{3\pi}\left[\frac{5}{3}\langle\bar{q}q\rangle - \langle\bar{s}s\rangle\right]\ln\frac{-q^2}{\mu^2} + \frac{m_q\alpha_s}{3\pi}\left[\frac{5}{3}\langle\bar{s}s\rangle - \langle\bar{q}q\rangle\right]\ln\frac{-q^2}{\mu^2} \\
&\quad - \left(\frac{m_q}{12} + \frac{m_s}{4}\right)\frac{\langle\bar{q}\sigma gGq\rangle}{q^2} + \left(\frac{m_s}{12} + \frac{m_q}{4}\right)\frac{\langle\bar{s}\sigma gGs\rangle}{q^2} + \frac{m_q m_s}{8q^2}\left\langle\frac{\alpha_s}{\pi}G^2\right\rangle\ln\frac{m_s^2}{m_q^2} \\
&\quad - \frac{m_s^2}{24q^2}\left\langle\frac{\alpha_s}{\pi}G^2\right\rangle\left[1 + \ln\frac{-q^2}{m_s^2}\right] + \frac{m_q^2}{24q^2}\left\langle\frac{\alpha_s}{\pi}G^2\right\rangle\left[1 + \ln\frac{-q^2}{m_q^2}\right] \\
&\quad - \frac{8\pi\alpha_s}{27q^2}\left[\langle\bar{q}q\rangle^2 - \langle\bar{s}s\rangle^2\right],
\end{aligned} \tag{A.5}$$

$$\begin{aligned}
\Pi^{(a)}_{G,2}(q^2) &= (m_s^2 - m_q^2)\,\frac{\alpha_s}{72\pi^3}\left[-5\ln^2\frac{-q^2}{\mu^2} + 25\ln\frac{-q^2}{\mu^2}\right] \\
&\quad + \frac{2m_s\alpha_s}{9\pi q^2}\langle\bar{q}q\rangle\left[\frac{1}{3} + \ln\frac{-q^2}{\mu^2}\right] - \frac{10m_s\alpha_s}{9\pi q^2}\langle\bar{s}s\rangle \\
&\quad - \frac{2m_q\alpha_s}{9\pi q^2}\langle\bar{s}s\rangle\left[\frac{1}{3} + \ln\frac{-q^2}{\mu^2}\right] + \frac{10m_q\alpha_s}{9\pi q^2}\langle\bar{q}q\rangle + \frac{m_s}{6q^4}\langle\bar{s}\sigma gGs\rangle - \frac{m_q}{6q^4}\langle\bar{q}\sigma gGq\rangle \\
&\quad + \frac{m_s^2}{6q^4}\left\langle\frac{\alpha_s}{\pi}G^2\right\rangle\left[1 - \frac{1}{2}\ln\frac{-q^2}{m_s^2}\right] - \frac{m_q^2}{6q^4}\left\langle\frac{\alpha_s}{\pi}G^2\right\rangle\left[1 - \frac{1}{2}\ln\frac{-q^2}{m_q^2}\right] \\
&\quad + \frac{8\pi\alpha_s}{27q^4}\left[\langle\bar{q}q\rangle^2 - \langle\bar{s}s\rangle^2\right].
\end{aligned} \tag{A.6}$$



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The expression for $\Pi_G^{(p)}$ has already been given in ref. [6], together with $\Pi_{G,(1,2)}^{(a)}$, to leading order in SU(3) breaking. The terms in $m_s^2$ and $m_q$ are new. The above expressions fulfill the relation (A.4).

At this point a few comments are in order concerning the structure of these formulas. The reader may have noticed that the Wilson coefficient of the gluon condensate contributions to the above correlation functions contain infrared sensitive terms $\sim \ln(-q^2/m_{q,s}^2)$. These terms appear to violate the structure of the operator product expansion which stipulates that long- and short-distance contributions be properly factorised and all long-distance contributions be absorbed into the condensates, leaving purely short-distance Wilson coefficients which must be analytic in $m_{q,s}$. As discussed in ref. [20], the appearance of terms logarithmic in $m_{q,s}$ is due to the fact that the above expressions are obtained using Wick's theorem to calculate the condensate contributions, which implies that the condensates are normal-ordered: $\langle O\rangle = \langle 0| : O : |0\rangle$. Recasting the OPE in terms of non-normal-ordered operators, all infrared sensitive terms can be absorbed into the corresponding condensates. Indeed, using [20]

$$\langle 0|\bar{s}gGs|0\rangle = \langle 0|:\bar{s}gGs:|0\rangle + \frac{m_s}{2}\,\log\frac{m_s^2}{\mu^2}\,\langle 0|:\frac{\alpha_s}{\pi}\,G^2:|0\rangle,$$

and the corresponding formula for $q$ quarks, all terms $\sim \ln m_{q,s}^2$ can be absorbed into the mixed quark-quark-gluon condensate and the resulting Wilson-coefficients can be expanded in powers of $m_{q,s}^2$. In calculating the sum rules, we hence will use

$$\ln\frac{-q^2}{m_{q,s}^2} \to \ln\frac{-q^2}{\mu^2}\,.$$

As for the structure of the ultraviolet logarithms $\sim \ln(-q^2/\mu^2)$, they follow from the mixing of the gluonic operator $\bar{q}(gG_{\alpha\mu})i\gamma^\mu\gamma_5 s$ with various quark-bilinear operators as given in eq. (20) in ref. [6].

### A.2 $\kappa_4^{\parallel}(K^*)$

The correlation functions used to determine $\kappa_4^{\parallel}(K^*)$ are very similar to those in the previous subsection. We choose $\Gamma_1^\mu = i\gamma^\mu$ and $\Gamma_2 = \sigma_{\beta\gamma}$ to obtain the pure-parity correlation function

$$\Pi_{\alpha\beta\gamma}(q) = i(g_{\alpha\beta}q_\gamma - g_{\alpha\gamma}q_\beta)\Pi_G^{(\sigma)}(q^2) \tag{A.7}$$

and $\Gamma_2 = \gamma_\beta$ which yields

$$\Pi_{\alpha\beta}(q) = g_{\alpha\beta}\Pi_{G,1}^{(v)}(q^2) + q_\alpha q_\beta \Pi_{G,2}^{(v)}(q^2). \tag{A.8}$$

$\Pi_G^{(\sigma)}$ and $\Pi_{G,1}^{(v)}$ receive contributions from $1^-$ states only and $\Pi_{G,2}^{(v)}$ from both $1^-$ and $0^+$ states. Another possible choice is $\Gamma_2 = \mathbb{1}$ which yields the pure-parity correlation function

$$\Pi_\alpha(q) = q_\alpha \Pi_G^{(s)}(q^2) \tag{A.9}$$



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with contributions from only $0^+$ states. $\Pi_G^{(s)}$ and $\Pi_{G,1(2)}^{(v)}$ are related by the equation of motion for the vector current:

$$\Pi_{G,1}^{(v)}(q^2) + q^2\Pi_{G,2}^{(v)}(q^2) = (m_s - m_q)\Pi_G^{(s)} + \text{ contact terms.} \tag{A.10}$$

The expression for $\Pi_G^{(\sigma)}$ was given in ref. [6], the other correlation functions are obtained by the simple replacements

$$\Pi_{G,1(2)}^{(v)}(q^2) \;=\; \Pi_{G,1(2)}^{(a)}(q^2)\Big|_{m_q\to -m_q,\,\langle\bar{q}q\rangle\to-\langle\bar{q}q\rangle,\,\langle\bar{q}\sigma gGq\rangle\to-\langle\bar{q}\sigma gGq\rangle} \tag{A.11}$$

$$\Pi_G^{(s)}(q^2) \;=\; \Pi_G^{(p)}(q^2)\Big|_{m_q\to -m_q,\,\langle\bar{q}q\rangle\to-\langle\bar{q}q\rangle,\,\langle\bar{q}\sigma gGq\rangle\to-\langle\bar{q}\sigma gGq\rangle} \tag{A.12}$$

which follows from the chiral structure of the correlation functions.

### A.3 $\kappa_4^\perp(K^*)$

For $\kappa_4^\perp(K^*)$, $\Gamma_1^\mu$ is given by $\sigma^{\beta\mu}$ and for $\Gamma_2$ we choose $\sigma_{\gamma\delta}$. The resulting correlation function has contributions from both $1^-$ and $1^+$ states and can be written as

$$\Pi_{\alpha\beta\gamma\delta}(q) = i\Pi_{G,4}^{1^-}(q^2)P^{1^-}_{4,\alpha\beta\gamma\delta} + i\Pi_{G,3}^{1^-}(q^2)P^{1^-}_{3,\alpha\beta\gamma\delta} + i\Pi_{G,4}^{1^+}(q^2)P^{1^+}_{4,\alpha\beta\gamma\delta}, \tag{A.13}$$

where the projectors $P^{1^\pm}$ are given by

$$\begin{aligned}
P^{1^-}_{4,\alpha\beta\gamma\delta} &= \frac{1}{q^2}\left[(g_{\alpha\gamma}q_\beta q_\delta - \{\alpha\leftrightarrow\beta\}) - (\{\gamma\leftrightarrow\delta\})\right],\\
P^{1^-}_{3,\alpha\beta\gamma\delta} &= \frac{1}{q^2}\left[(g_{\alpha\gamma}q_\beta q_\delta + \{\alpha\leftrightarrow\beta\}) - (\{\gamma\leftrightarrow\delta\})\right],\\
P^{1^+}_{4,\alpha\beta\gamma\delta} &= \frac{1}{q^2}\left[P^{1^-}_{4,\alpha\beta\gamma\delta} + q^2 g_{\beta\gamma}g_{\alpha\delta} - q^2 g_{\alpha\gamma}g_{\beta\delta}\right].
\end{aligned}$$

$P_3^{1^-}$ projects onto the twist-3 matrix element $\kappa_3^\perp(K^*)$, $P_4^{1^-}$ onto $\kappa_4^\perp(K^*)$ and $P_4^{1^+}$ onto the contribution from $1^+$ intermediate states. As usual, $\Pi_{\alpha\beta\gamma\delta}$ must not have a singularity at $q^2 = 0$ which implies

$$\Pi_{G,4}^{1^-}(0) + \Pi_{G,4}^{1^+}(0) = 0.$$

That means that one can construct a mixed-parity sum rule from $\Pi_{G,4} \equiv (\Pi_{G,4}^{1^-}(q^2) + \Pi_{G,4}^{1^+}(q^2))/q^2$ which has lower dimension than the pure-parity sum rule obtained from $\Pi_{G,4}^{1^-}$ alone. We find[5]

$$\begin{aligned}
\Pi_{G,4}^{1^-}(q^2) = {}& (m_s^2 - m_q^2)\,\frac{\alpha_s}{144\pi^3}\,q^2\left[3\ln^2\frac{-q^2}{\mu^2} - 11\ln\frac{-q^2}{\mu^2}\right]\\
&+ \frac{\alpha_s m_s}{3\pi}\left[\frac{5}{6}\langle\bar{s}s\rangle + \left(\ln\frac{-q^2}{\mu^2} - \frac{5}{3}\right)\langle\bar{q}q\rangle\right] - \frac{\alpha_s m_q}{3\pi}\left[\frac{5}{6}\langle\bar{q}q\rangle + \left(\ln\frac{-q^2}{\mu^2} - \frac{5}{3}\right)\langle\bar{s}s\rangle\right]\\
&+ \frac{1}{12q^2}\langle\bar{q}\sigma gGq\rangle(2m_s + m_q) - \frac{1}{12q^2}\langle\bar{s}\sigma gGs\rangle(m_s + 2m_q)
\end{aligned}$$

[5]We also give $q^2$-independent terms in the quark condensate contribution to $\Pi_{G,4}^{1^\pm}$ because they are needed for calculating $\Pi_{G,4}$. Note that for $\Pi_{G,4}^{1^\pm}$ these terms are affected by finite counterterms as discussed in ref. [7], which however cancel in the sum $\Pi_{G,4}^{1^-} + \Pi_{G,4}^{1^+}$.

$$
+\frac{1}{24q^2}\left\langle\frac{\alpha_s}{\pi}G^2\right\rangle\left\{-m_s^2\left[2-\ln\frac{-q^2}{m_s^2}\right]+m_q^2\left[2-\ln\frac{-q^2}{m_q^2}\right]+2m_qm_s\ln\frac{m_q^2}{m_s^2}\right\}
$$
$$
+0\cdot(\langle\bar qq\rangle^2-\langle\bar ss\rangle^2), \tag{A.14}
$$

$$
\begin{aligned}
\Pi_{G,4}^{1^+}(q^2) &= (m_s^2-m_q^2)\,\frac{\alpha_s}{144\pi^3}\,q^2\left[3\ln^2\frac{-q^2}{\mu^2}-17\ln\frac{-q^2}{\mu^2}\right]\\
&+\frac{\alpha_s m_s}{3\pi}\left[\frac{7}{6}\langle\bar ss\rangle+\left(-\ln\frac{-q^2}{\mu^2}+\frac{4}{3}\right)\langle\bar qq\rangle\right]-\frac{\alpha_s m_q}{3\pi}\left[\frac{7}{6}\langle\bar qq\rangle+\left(-\ln\frac{-q^2}{\mu^2}+\frac{4}{3}\right)\langle\bar ss\rangle\right]\\
&+\frac{1}{12q^2}\langle\bar q\sigma gGq\rangle(2m_s-m_q)-\frac{1}{12q^2}\langle\bar s\sigma gGs\rangle(m_s-2m_q)\\
&+\frac{1}{24q^2}\left\langle\frac{\alpha_s}{\pi}G^2\right\rangle\left\{-m_s^2\left[2-\ln\frac{-q^2}{m_s^2}\right]+m_q^2\left[2-\ln\frac{-q^2}{m_q^2}\right]-2m_qm_s\ln\frac{m_q^2}{m_s^2}\right\}\\
&+0\cdot(\langle\bar qq\rangle^2-\langle\bar ss\rangle^2),
\end{aligned}\tag{A.15}
$$

$$
\begin{aligned}
\Pi_{G,4}(q^2) &= (m_s^2-m_q^2)\,\frac{\alpha_s}{72\pi^3}\left[3\ln^2\frac{-q^2}{\mu^2}-14\ln\frac{-q^2}{\mu^2}\right]\\
&+\frac{\alpha_s m_s}{9\pi q^2}\left[6\langle\bar ss\rangle-\langle\bar qq\rangle\right]-\frac{\alpha_s m_q}{9\pi q^2}\left[6\langle\bar qq\rangle-\langle\bar ss\rangle\right]\\
&+\frac{1}{6q^4}\left(m_q\langle\bar q\sigma gGq\rangle-m_s\langle\bar s\sigma gGs\rangle\right)\\
&+\frac{1}{12q^4}\left\langle\frac{\alpha_s}{\pi}G^2\right\rangle\left\{-m_s^2\left[2-\ln\frac{-q^2}{m_s^2}\right]+m_q^2\left[2-\ln\frac{-q^2}{m_q^2}\right]\right\}.
\end{aligned}\tag{A.16}
$$

## B. Borel transforms

QCD sum rules are obtained from the Borel transforms of the correlation functions listed in the previous section. Most of the transforms are straightforward, except for those of expressions of type $1/(q^2)^n\ln(-q^2/\mu^2)$, which can, however, be conveniently calculated using the formula

$$
\frac{1}{\pi}\,\mathrm{Im}(-q^2-i0)^\alpha=\frac{s^\alpha}{\Gamma(-\alpha)\Gamma(1+\alpha)}\,\Theta(s)
$$

with $s=-q^2$. We then obtain, including continuum subtraction of contributions from $s>s_0$,

$$
\begin{aligned}
\mathcal{B}_{M^2}^{\rm sub}\,\frac{1}{q^2}\,\ln\frac{-q^2}{\mu^2} &= \gamma_E-\ln\frac{M^2}{\mu^2}+\int_{s_0}^\infty\frac{ds}{s}\,e^{-s/M^2},\\
\mathcal{B}_{M^2}^{\rm sub}\,\frac{1}{(q^2)^2}\,\ln\frac{-q^2}{\mu^2} &= \frac{1}{M^2}\left(1-\gamma_E+\ln\frac{M^2}{\mu^2}+M^2\int_{s_0}^\infty\frac{ds}{s^2}\,e^{-s/M^2}\right).
\end{aligned}
$$

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